Can a Mixed Guidance Strategy Improve Missile Performance?

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The terminal phase of a missile vs aircraft engagement in an uncertain, noise-corrupted environment is formulated as a partial-information differential game. The cost function of the game is the missile's single-shot kill probability. A new approach, allowing a mixed missile guidance strategy, leads to a feasible saddle-point solution and circumvents the difficulties encountered in previous investigations. A simple example demonstrates that in some cases, the implementation of optimal mixed strategies is indeed advantageous for both parties. The present paper is the first step in the search for feasible saddle-point solutions for a large class of nondeterministic pursuit-evasion games.

Introduction

IN the past two decades, many papers dealing with different aspects of the terminal phase of missile-aircraft encounters have been published. Some papers formulate this encounter as a one-sided guidance[1-8] or avoidance[9-13] problem. Others[14-23] formulate it as a two-person differential game, be it in the deterministic setting or in the stochastic setting.

The stochastic differential game approach to the problem has yet to yield satisfactory solutions. It was formally shown[18] that the optimal strategies of both players in the stochastic game require infinite dimensional controllers. Different approaches to overcoming the infinite dimensionality problem were taken by several authors.[19-23] In Ref. 20, finite-dimensional strategies are proposed which seem to be computationally intractable in practical cases. In Ref. 19, the pursuit-evasion problem is formulated as a nonzero sum game and complexity-constrained Nash strategies are found. In general, these optimal Nash strategies are not security strategies. They do not guarantee an optimal outcome unless both players are irrevocably committed to them. In Refs. 21–23, as in Ref. 19, the problem is formulated as a nonzero sum game, but instead of searching for Nash equilibria, the authors search for the security strategies of both players. The security strategies turn out to be of finite dimension and feasible. In many cases, however, the guaranteed "worst-case" outcome (loss ceiling or gain floor) may not be acceptable. All of the previously mentioned papers[18-23] share the assumption that the optimal strategies of both players are pure.

A different approach to solving the problem is based on the assumption that the optimal strategies are mixed.‡ This approach follows the comment made by Isaacs in his book:[25] "One of the main difficulties in differential games with incomplete information is that undoubtedly optimal play will, in all essential cases, require mixed strategies."

There are several previous works on mixed strategies in differential games.[26-29] In Refs. 26–28, it is shown that the optimal strategy of a tracked target, which inflicts maximum estimation errors on the tracking radar, is mixed. In Ref. 29, a pursuit-evasion game with a finite detection range is presented, showing that under certain circumstances, the optimal midcourse strategies of both players are mixed.

The random nature of the evader's optimal strategy in the terminal phase of a missile-aircraft scenario is well established and can be logically deduced from the results of Refs. 14, 26, and 28. It can be seen from Ref. 14 that if the information available to the missile is perfect, then for practical values of acceleration and speed ratios between the missile and the aircraft, the guaranteed miss distance is so small that it renders ineffective all evasive maneuvers. This result leads to the conclusion that the aircraft's attention should be focused on information denial. Since the information for missile guidance is generally extracted from an estimator, the aircraft's objective should be to generate maximum estimation errors. This can be achieved only by applying a mixed strategy, as concluded in Refs. 26 and 28. The intuitive interpretation of this result is that the random nature of the evader's mixed-strategy structure actually reduces the a priori information available for estimation and thus deteriorates the quality of the estimates obtainable by the pursuer.

However, to the authors' best knowledge, no attempt has been made in the open literature to consider optimal mixed strategies for the pursuer in the terminal phase of a missile-aircraft scenario. The objective of this paper is to take the first step in this direction by introducing a new formulation of the problem which allows mixed strategies for both players. This formulation presents an entirely new approach in guided missile design and opens the way for determining the optimal mixed strategies for the missile. The paper outlines this new problem formulation and presents a simple illustrative example that answers affirmatively the question posed by the title.

Formulation of the Problem

In this section, the terminal phase of a future missile-aircraft encounter is formulated as a two-dimensional, two-person, zero-sum, imperfect-information, linear (not quadratic) differential game of predetermined duration in which the allowable strategies of both players are mixed. The players are 1) the missile or the agent that fires it and 2) the pilot of the evading aircraft. For the sake of simplicity, we shall refer to them in the sequel as "the pursuer" and "the evader," respectively, and it will be understood that "the pursuer" also stands for the agent that fires the missile and that "the evader" includes the aircraft and its pilot, who makes the decisions.

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‡A mixed strategy is a probability distribution on a pure-strategy set.[24]

In the formulation of the problem, the ECM capability of the evader, which may be necessary to enhance aircraft survivability, is taken into account in the form of "electronic jinking."[13] This is a method which electronically generates an apparent motion of the aircraft's radar reflection center.

The information structure of the game is taken to be as follows:

1) The pursuer has exact a priori knowledge of the duration of the game.

2) The pursuer measures the evader's position perpendicular to a reference line throughout the duration of the game. This measurement is corrupted by noise.

3) The evader knows when the game starts but has no knowledge as to the exact duration of the game.

4) Throughout the game, no measurements concerning the state of the game are available to the evader.

Assuming that the game takes place in the vicinity of a collision course (see Fig. 1), the linearized kinematics point-mass equations of motion can be written as (see the Appendix)

$$\dot{x} = Ax + Bu + Cv \tag{1}$$

where the state x is an n vector and u and v are scalar control variables. In the Appendix, x, A, B, and C are defined. The single control variable available to the pursuer, $u \in U$, represents the commanded acceleration perpendicular to the inertial reference line (see Fig. 1). One of the two control variables available to the evader, $v \in V$, represents its acceleration perpendicular to the inertial reference line.

The pursuer's measurements z are given by

$$z = H_1 X + H_2(w + \zeta) \tag{2}$$

where z is a k vector $(k \leq n-1)$ and w and ζ are scalars. The matrices H_1 and H_2 are defined in the Appendix. The second control variable available to the evader, $w \in W$, represents an intentionally introduced position disturbance, i.e., electronic jinking. The zero-mean white Gaussian measurement noise ζ has covariance $R_\zeta(\tau) = \phi_\zeta \delta(\tau)$.

The admissible control sets U, V, and W are defined by

$$U = \{ u(t);\, |u(t)| \leq a_{p\,\max} |\cos\gamma_{po}|\, \forall t \in [0, t_f] \}$$

$$V = \{ v(t);\, |v(t)| \leq a_{e\,\max} |\cos\gamma_{eo}|,\, |\dot{v}(t)| \leq \alpha_L\, \forall t \in [0, t_f] \}$$

$$W = \{ w(t);\, |w(t)| \leq w_{\max}\, \forall t \in [0, t_f] \} \tag{3}$$

where $a_{p\,\max}$ and $a_{e\,\max}$ are the pursuer's and evader's maximum lateral accelerations, respectively, $w_{\max}$ is the maximum possible disturbance due to the electronic jinking, and α_L is a parameter by which the roll dynamics of the evading aircraft are indirectly accounted for. The nominal collision course flight-path angles are γ_{po} and γ_{eo} (see Fig. 1).

The payoff of the game, J, is the "single-shot kill probability" (SSKP), which is defined as

$$J = \text{SSKP} = E\{ P[x_1(t_f)] \} \tag{4}$$

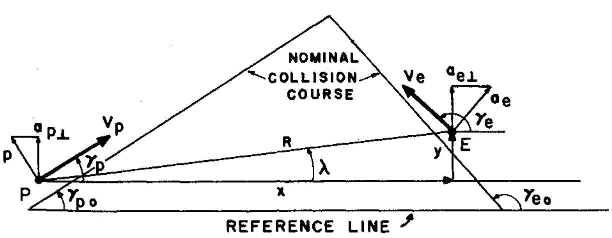

Fig. 1 Relative geometry of the game.

where $P[\cdot]$ is a real-valued function which describes the warhead lethality and which is subject to $0 \leq P[\cdot] \leq 1$. The pursuer wishes to maximize J while the evader wishes to minimize it. This payoff function, which is indeed the one of true practical interest, has not been used in previous works.

The pure-strategy set of the pursuer, Δ_p, is defined as a countable set of "guidance policies" of a predetermined structure. A "guidance policy" is understood to be composed of a guidance law and an estimator. The different guidance policies in Δ_p result from different assumptions made on the target maneuver model. In general, each assumption leads to a different guidance law and estimator. More explicitly, assuming m_p different target maneuver models, Δ_p is given by

$$\Delta_p = \{ \delta_{p_j},\, j = 1,2,\ldots, m_p \} \tag{5}$$

where each pure strategy δ_{p_j} is of the form

$$\delta_{p_j} = a_{p_{\max}} |\cos\gamma_{p0}|\, sat\left(\frac{g_j(\hat{x}^{(j)})}{a_{p_{\max}} |\cos\gamma_{p0}|} \right) \tag{6}$$

where g_j and $\hat{x}^{(j)}$ are the jth guidance law and output of the jth estimator, respectively.

Equation (6) simply states that δ_{p_j} is a mapping, subject to some constraints, from the estimated state space to the control space. It is a generalized form for the guidance policies considered, also covering proportional navigation and other guidance policies discussed in previous works.[3,5,7,8,14]

As far as the general formulation of the problem is concerned, there is no need at this point to specify any further the detailed structure of the guidance laws and estimators. It should be pointed out, however, that in any practical attempt to actually solve the problem, these structures will have to be determined in a fairly specific and detailed form.

The pure-strategy set of the evader, Δ_e, is defined as a countable set of "actions" δ_{ei}, each of which is composed of a maneuver sequence and an electronic counter-measures policy. In other words,

$$\Delta_e = \{ \delta_{ei},\, i = 1,2,\ldots, m_e \} \tag{7}$$

where each pure strategy δ_{ei} is defined by the pair

$$\delta_{ei} = [v_i(t), w_i(t)] \qquad v_i(t) \in V,\, w_i(t) \in W \tag{8}$$

Note that both m_p and m_e in Eqs. (5) and (8) may be infinite.

The game is played as follows: at the beginning of the game, or shortly prior to it, each player "chooses," through a chance mechanism, one of its pure strategies and plays accordingly until the end of the game. The chance mechanism, which determines the pure strategy to be played, is a mechanization of the player's mixed strategy. A generalized block diagram of the pursuer's guidance loop which corresponds to the mixed strategy concept is given in Fig. 2.

It is important to emphasize that, due to the short duration of the missile aircraft engagement, the selection of one of the available pure strategies at the beginning of the game seems to be the only reasonable way to act in a future operational scenario. This statement is based on the following:

1) It is practically impossible for the evading aircraft to determine with reasonable accuracy in real time the guidance policy used by the missile in a particular encounter, even by taking measurements of its flight path. Thus, once the evading pilot has "chosen" his strategy, there is no reason to change it throughout the engagement.

2) The estimation of a random (unpredictable) evasive motion, based on noisy measurements of the displacements only, is known to be a very difficult task (see example 2, Chap. 14, Ref. 30). It is even more difficult to make an estimate of the model or parameters which best describe the actual maneuver. Thus, as long as the evading aircraft maneuvers in

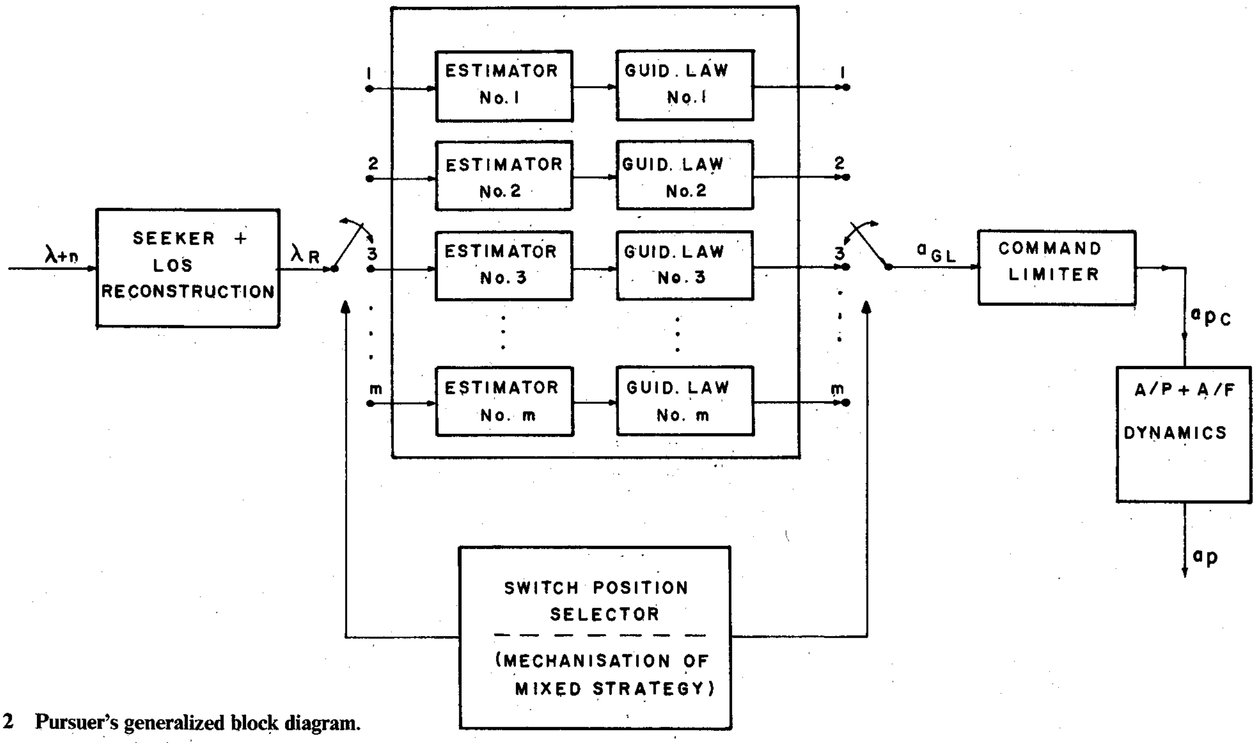

Fig. 2 Pursuer's generalized block diagram.

an unpredictable way, it is clear that the missile is unable to use its measurements for verifying in real time the assumption on which its estimator is based. This means that throughout the duration of the engagement, there is no "basis" for changing the guidance policy of the missile. Note that the selection of one of the pure strategies at the outset actually transforms the problem into a matrix game.

The evader's and pursuer's mixed strategies are determined by sequences of real numbers, $\{\alpha_i\}_{i=1,\ldots,m_e}$ and $\{\beta_j\}_{j=1,\ldots,m_p}$, respectively, which satisfy

$$\sum_{i=1}^{m_e} \alpha_i = 1, \qquad \alpha_i \geq 0 \quad \forall i = 1,2,\ldots,m_e$$

$$\sum_{j=1}^{m_p} \beta_j = 1, \qquad \beta_j \geq 0 \quad \forall j = 1,2,\ldots,m_p \tag{9}$$

where α_i determines the probability of "choosing" δ_{ei} by the evader and β_j determines the probability of "choosing" δ_{pj} by the pursuer. The payoff function, Eq. (4), in terms of Δ_e, Δ_p, $\{\alpha_i\}$, and $\{\beta_j\}$ is given by

$$J = J(\Delta_e, \{\alpha_i\}, \Delta_p, \{\beta_j\}) = \sum_{i=1}^{m_e}\sum_{j=1}^{m_p} \alpha_i \beta_j P_{ij}$$

$$\triangleq J_{\Delta_e,\Delta_p}(\{\alpha_i\},\{\beta_j\}) \tag{10}$$

where P_{ij} is the SSKP for the case in which the pure strategies δ_{ei} and δ_{pj} are played, and it can be expressed by

$$P_{ij} = E\{P[x_1(t_f)] | \delta_{e_i}, \delta_{p_j}\} \tag{11}$$

Given the pure-strategy sets Δ_e and Δ_p, the solution of the game is presented by a triplet: the optimal sequences $\{\alpha_i^*\}\{\beta_j^*\}$, called the optimal mixed strategies of the evader and the pursuer, respectively, and a real number $0 \leq V_m \leq 1$, which is called the value of the game. The definition of V_m is

$$V_m = J_{\Delta_e,\Delta_p}(\{\alpha_i^*\},\{\beta_j^*\}) \tag{12}$$

The value satisfies a saddle-point inequality

$$J_{\Delta_e,\Delta_p}(\{\alpha_i^*\},\{\beta_j\}) \leq V_m \leq J_{\Delta_e,\Delta_p}(\{\alpha_i\},\{\beta_j^*\}) \tag{13}$$

for every arbitrary sequence $\{\alpha_i\}$ or $\{\beta_j\}$ satisfying Eq. (9).

Obviously, $\{\alpha_i^*\}$, $\{\beta_j^*\}$, and V_m are functions of Δ_e and Δ_p. Thus,

$$\{\alpha_i^*\} = \{\alpha_i^*(\Delta_e,\Delta_p)\}, \qquad \{\beta_j^*\} = \{\beta_j^*(\Delta_e,\Delta_p)\}$$

$$V_m = V_m(\Delta_e,\Delta_p) \tag{14}$$

Generalized Problem Formulation

Motivated by Eq. (14), the generalized problem is formulated as follows:

Given an imperfect information pursuit-evasion game in which both the pursuer and the evader "select" at the outset a strategy from pure-strategy sets Δ_e and Δ_p of the form of Eqs. (5), (6) and (7), (8), respectively, and in which the payoff function is the single-shot kill probability given by Eq. (4), find the optimal pure strategy sets Δ_e^* and Δ_p^* that satisfy the following saddle-point relationship:

$$V_m(\Delta_e^*,\Delta_p) \leq V_m(\Delta_e^*,\Delta_p^*) \leq V_m(\Delta_e,\Delta_p^*) \tag{15}$$

for every admissible Δ_e and Δ_p.

An Illustrative Example

The objectives of this example are rather modest. It does not present the solution of the generalized problem. It only shows that under certain circumstances, the optimal strategy of the pursuer is indeed a mixed one. The example also intends to give the *motivation* for solving the generalized problem—to find the mathematical conditions for the selection of the optimal pure-strategy sets and to investigate the conditions under which the optimal strategy of the pursuer is truly mixed.

For the sake of simplicity, it is assumed in this example that the evading aircraft is not equipped with an electronic counter measure (ECM) system and that the missile's autopilot response can be represented by a first-order time constant, τ_p. Also, it is assumed that the pure-strategy sets Δ_e and Δ_p are given in advance. These sets are not intended to serve as an approximation of the optimal sets Δ_e^* and Δ_p^*, but are chosen arbitrarily on a heuristic basis motivated by Refs. 3–5, 7, 8, 11, and 12.

The evader's pure-strategy set Δ_e is assumed to be composed of only four elements, namely, a constant maximum lateral acceleration maneuver and three sinusoidal random phase maneuvers at some given frequency. Explicitly,

$$\delta_{e1} = [\, a_{e\,\max}|\cos\gamma_{eo}|, 0] \tag{16a}$$

$$\delta_{ei} = [\, a_{e\,\max}|\cos\gamma_{eo}|\sin(\omega_i t + \phi), 0] \qquad i = 2,3,4 \tag{16b}$$

where $\omega_i = i - 1$ and ϕ is the uniformly distributed random phase over $[0, 2\pi]$. (Note that δ_{e_1} can also be expressed as a sinusoidal maneuver at zero frequency and at phase $\pi/2$.)

The pursuer's pure-strategy set Δ_p is assumed to be composed of six guidance policies of the form of Eq. (6). Proportional navigation with $N' = 3.5$ is δ_{p_1} (used for comparison purposes), and the other five guidance policies are of the following structure:

$$g_j(\cdot) = \frac{N'(t_{go})}{t_{go}^2} ZEM^{(j)} \tag{17}$$

where $N'(t_{go})$ is the time-varying effective navigation gain, $ZEM^{(j)}$ is the "zero-effort-miss," and t_{go} is the time-to-go. $N'(t_{go})$ and $ZEM^{(j)}$ are given by:[5]

$$N'(t_{go}) = \frac{1}{t_{go}^2} \frac{6T^2(T - 1 + e^{-T})}{3 + 6T - 6T^2 + 2T^3 - 12Te^{-T} - 3e^{-2T}} \tag{18}$$

$$ZEM^{(j)} = \hat{y}^{(j)} + t_{go}\hat{\dot{y}}^{(j)} - \tau_p^2(T - 1 + e^{-T})\, a_p|\cos\gamma_{po}| + (\Delta ZEM)_e^{(j)} \tag{19}$$

with

$$T = t_{go}/\tau_p \tag{20}$$

where $\hat{y}$ is the estimated relative-position perpendicular to the line of sight, a_p is the pursuer's lateral acceleration, and $(\Delta ZEM)_e^{(j)}$ is the contribution of the evader's maneuver to the "zero-effort-miss." The shown form of the guidance law is optimal for a given evader's maneuver in the unconstrained deterministic case.[3,5,7,8]

We chose δ_{p2} to be a "wide-band guidance policy" which achieves similar miss distances (in the rms sense) against all δ_{e_i}. We chose δ_{p3}–δ_{p6} to be "narrow-band guidance policies" designed to perform "optimally" only against a specified δ_{e_i}.

For δ_{p2} and δ_{p3}, $(\Delta ZEM)_e^{(j)}$ is given by

$$(\Delta ZEM)_e^{(j)} = \frac{k \cdot t_{go} - 1 + e^{-kt_{go}}}{k^2} \hat{a}_{e\perp}^{(j)}, \qquad j = 2,3 \tag{21}$$

where $a_{e\perp}$ is the estimated evader's acceleration perpendicular to the line of sight. For δ_{p2}, $k = 2$, and for δ_{p3}, $k = 0.2$. For δ_{p4}–δ_{p6}, $(\Delta ZEM)_e^{(j)}$ is given by

$$(\Delta ZEM)_e^{(j)} = \frac{1}{\omega_j^2}(1 - \cos\omega_j t_{go})\, \hat{a}_{e\perp}^{(j)} + \frac{1}{\omega_j^3}(\omega_j t_{go} - \sin\omega_j t_{go})\, \hat{\dot{a}}_{e\perp}^{(j)} \tag{22}$$

where $j = 4,5,6$ and $\omega_j = (j - 3)$ rad/s.

The structure and gains of the estimator corresponding to each δ_{p_j} are presented in Fig. 3 and Table 1, respectively. The estimators for δ_{p2}–δ_{p6} are steady-state Kalman filters based on the respective assumed target maneuvers.

For the specific parameter values presented in Table 2, the rms miss distances obtained for different pure-strategy combinations are presented in Fig. 4. These results are the average of 1000 Monte Carlo simulation runs for each point.

From Fig. 4, it is seen that δ_{p3}–δ_{p6} perform well when the actual maneuver encountered matches the design maneuver, but that they perform poorly when an off-design maneuver is encountered. In contrast, δ_{p2} performs equally well (in the rms sense) against all maneuvers encountered. The price paid for such uniform performance is a higher rms level of the minimal miss distances achieved. It is also seen that the performance of proportional navigation (with the specific parameters chosen) is similar to the performance of δ_{p3}.

In order to solve the problem at hand, one has to determine also the warhead lethality model. For the intended objectives of this example, a simplified lethality model is defined as

$$P[x_1(t_f)] = \begin{cases} 1 & \text{if } |x_1(t_f)| \le R_e \\ 0 & \text{otherwise} \end{cases} \tag{23}$$

where R_e represents the lethality radius. Since the pure-strategy sets of both players are finite, and since both players make their choices at the beginning of the game, the problem at hand is transformed, as mentioned earlier, into a standard two-person zero-sum finite-dimensional matrix game.[24]

Based on the results of the Monte Carlo simulation runs summarized in Fig. 4, the entries of the 6×4 payoff (SSKP) matrix can be computed for any value of lethality radius. In Table 3, the SSKP matrix for a lethal radius of 4 m is presented. It can be directly observed that the upper value (minmax) of the game is $\bar{V} = 0.75$, while the lower value (maxmin) is $\underline{V} = 0.32$. Since $\bar{V} \neq \underline{V}$, this matrix game has a solution only in mixed strategies.[24]

The optimal mixed-strategy solution of the matrix game was computed with a linear programming technique.[31] The value of the game in mixed strategies is $V_m(\Delta_e, \Delta_p) = 0.37$, and the optimal mixed strategies $\{\alpha_i^*(\Delta_e, \Delta_p)\}$ and $\{\beta_j^*(\Delta_e, \Delta_p)\}$ are given by

$$\{\alpha_i^*\} = (0.282, 0.251, 0.416, 0.051) \tag{24}$$

$$\{\beta_j^*\} = (0.003, 0.873, 0.110, 0.0, 0.015, 0.0) \tag{25}$$

The interpretation of the preceding results is the following: if both players were restricted to pure strategies, the pursuer's security strategy (δ_{p2} in Table 3) would guarantee a SSKP no greater than 0.32, while the evader's security strategy (δ_{e4} in Table 3) would guarantee a SSKP no smaller than 0.75. However, if both players are allowed to apply mixed strategies, the saddle-point SSKP that is guaranteed for *both parties* is 0.37.

In other terms, it can be said that in the present example, using mixed strategies increases the expected number of surviving aircraft by a ratio of 2.5. From the opposite point of view, it can be stated that the number of missiles needed to guarantee a kill probability of about 0.85 of a single airplane is reduced from 5 to 4 by the use of the optimal mixed strategy.

The origin of the demonstrated optimality of the mixed strategy is the explicit consideration of warhead lethality in evaluating the expected outcome of the missile vs aircraft pursuit-evasion game. This is an element which has been overlooked in all previous works, at least in the open literature. If warhead and proximity fuse designs can be assumed unconstrained, it is indeed true that there exists a pure strategy for the missile which guarantees high levels of SSKP for all possible target maneuvers. For the present example, a lethality range of 12 m would guarantee a SSKP of 0.93 with "δ_{p2}" as a unique missile guidance option. There may also be other

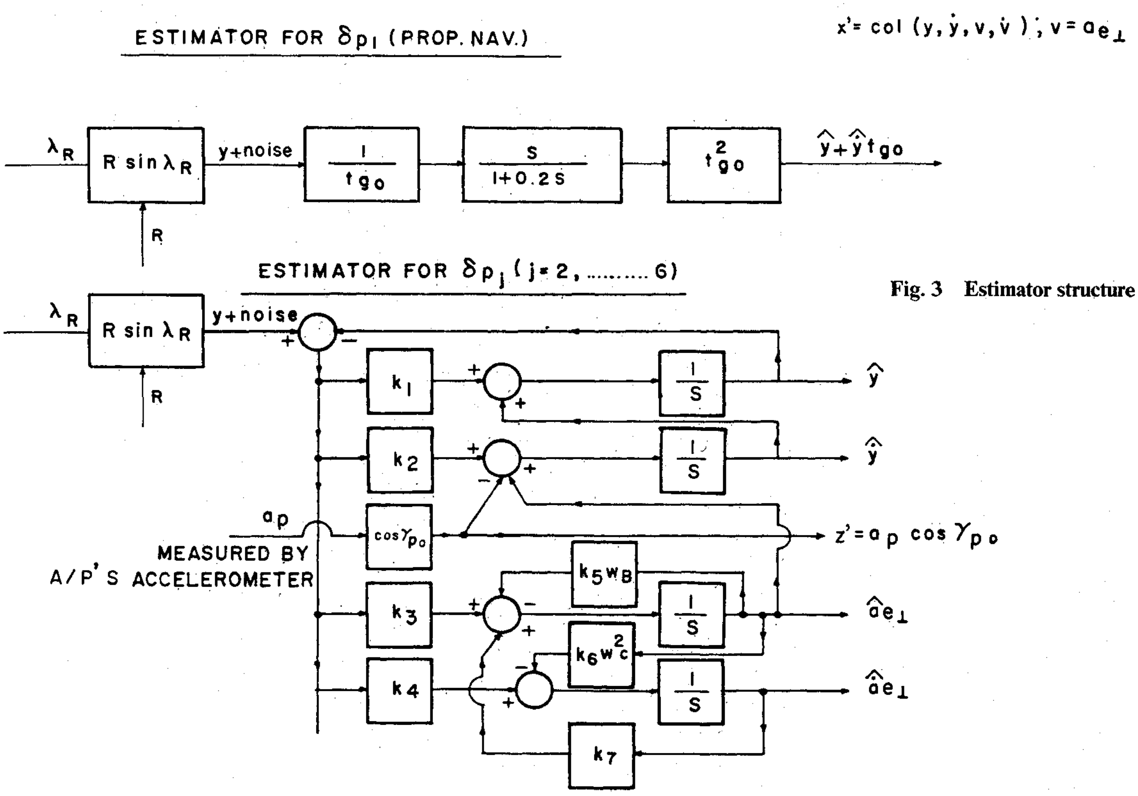

Fig. 3 Estimator structure for δ_{p_j}, $j = 1,2,\ldots,6$.

Table 1 Estimator gains for δ_{p_j}, $j = 2,\ldots,6$

j	2	3	4	5	6
k_1	10.35	6.13	6.59	8.09	8.81
k_2	53.54	18.78	21.72	32.73	38.79
k_3	116.06	27.87	39.62	66.46	73.51
k_4	0.0	0.0	25.28	2.00	−105.0
k_5	1.0	1.0	0.0	0.0	0.0
k_6	0.0	0.0	1.0	1.0	1.0
k_7	0.0	0.0	1.0	1.0	1.0
w_B	2.0	0.2	0.0	0.0	0.0
w_C	0.0	0.0	1.0	2.0	3.0

Table 2 Parameter values for the illustrative example (Fig. 4)

	Pursuer's dynamics ($q = 1$)
$t_f = 5$ s	$A_p = -1/\tau_p$
$\gamma_{po} = 0.0$ rad	$B_p = 1/\tau_p$
$\gamma_{eo} = \pi$ rad	$\tau_p = 0.2$ s
$a_{p\,\max} = 150$ m/s^2	*Measurement matrices* ($k = 2$)
$a_{e\,\max} = 50$ m/s^2	$H_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $H_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
$w_{\max} = 0.0$ m	
$\phi_\xi = 1$ m^2/Hz	

Table 3 Payoff (SSKP) matrix for the example

	$j \backslash i$	δ_{e_i} 1	2	3	4
δ_{p_j}	1	0.89	0.17	0.15	0.23
	2	0.32	0.38	0.40	0.40
	3	0.83	0.35	0.10	0.12
	4	0.06	0.76	0.06	0.09
	5	0.0	0.17	0.76	0.21
	6	0.0	0.12	0.31	0.75

W/H lethal range = 4 m.

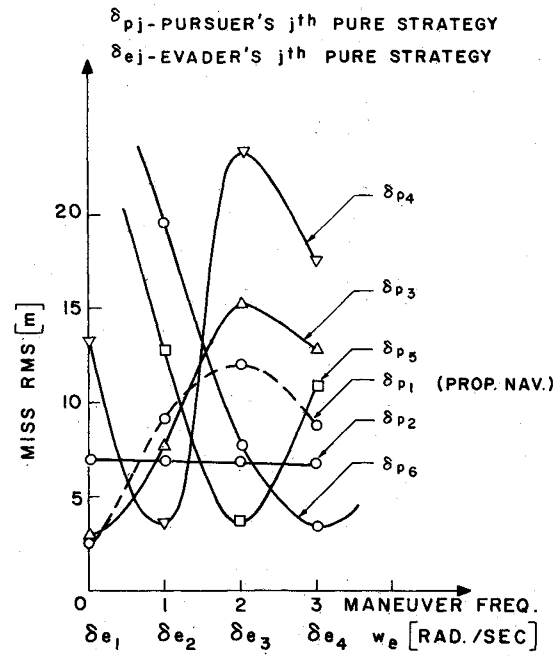

Fig. 4 Rms miss distances for the different pure-strategy combinations.

cases, depending on the parameters of the problem, for which the optimal mixed strategy of the pursuer will turn out to be pure.

Conclusions

This paper presents a *new formulation* of the terminal phase of a missile-aircraft encounter as a partial-information zero-sum differential game. The innovative features of this formulation are the following:

1) Both players are allowed to use mixed strategies.
2) The player's strategies are selected by a chance mecha-

nism at the outset of each encounter and remain fixed throughout that particular encounter.

3) The payoff function of the game is the missile's single-shot kill probability.

The generalized problem presented in the paper consists of searching for the optimal pure-strategy sets of the players satisfying a saddle-point inequality. The solution of this generalized problem is the subject of an extensive ongoing investigation.

In the limited scope of the present paper, only an illustrative example is given, which demonstrates that in certain cases, a mixed guidance strategy indeed improves missile performance. The new formulation and the eventual solution of the generalized problem have the potential of introducing an unconventional approach in future guided missile design, yielding improved performance in imperfect information scenarios.

Appendix

Derivation of Linearized Equations of Motion

Assumptions:

1) The game takes place in the horizontal plane.

2) Gravity is neglected.

3) The game takes place in the vicinity of the collision course (see Fig. 1).

4) The initial LOS between the pursuer and the evader is set as the inertial reference line.

5) The pursuer and the evader are point masses.

6) The pursuer's and evader's velocities parallel to the reference line are constant throughout the game.

7) The pursuer's and evader's accelerations perpendicular to the reference line are limited by $a_{p\,\max}|\cos\gamma_{po}|$ and $a_{e\,\max}|\cos\gamma_{eo}|$, respectively.

8) The pursuer's pitch dynamics are given by

$$\dot{x}_p = A_p x_p + B_p a_{pc}$$

where A_p and B_p are $q \times q$ and $q \times 1$ time invariant matrices. The commanded acceleration perpendicular to the velocity vector V_p is a_{pc}. The first element of x_p is a_p, the pursuer's actual acceleration perpendicular to V_p (see Fig. 1).

9) The evader's pitch dynamic response is neglected, and it is assumed that $a_{e\perp}$ is a control variable. (The roll dynamics are taken into account indirectly by limiting the rate of change of $a_{e\perp}$).

Derivation of the Equations

Let us define (see Fig. 1)

$$x_1 = y, \qquad x_2 = \dot{x}_1$$

$$v = a_{e\perp}, \qquad u = a_{po}|\cos\gamma_{po}| \tag{A1}$$

By making use of assumptions 1–6, it is easily seen that the equation of motion governing the game is given by (see Fig. 1)

$$\ddot{y} = a_{e\perp} - a_{p\perp} \tag{A2}$$

By making use of assumptions 7–9 and of Eq. (A1), Eq. (A2) becomes

$$\dot{x} = Ax + Bu + Cv \tag{A3}$$

where

$$x = col\left(x_1, x_2 \,|\, x_p^T |\cos\gamma_{po}|\right)$$

$$A = \left[\begin{array}{cc|ccc} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & -1 & 0 \; \cdots & 0 \\ \hline 0 & & & A_p & \end{array}\right],$$

$$B = \left[\begin{array}{c} 0 \\ 0 \\ \hline B_p \end{array}\right], \quad C = \left[\begin{array}{c} 0 \\ 1 \\ \hline 0 \end{array}\right] \tag{A4}$$

and where $|u| \le a_{p\,\max}|\cos\gamma_{po}|$, $|v| \le a_{e\,\max}|\cos\gamma_{eo}|$, and $|\dot{v}| \le \alpha_L$ (α_L is a parameter by which the roll dynamics of the evading aircraft are indirectly accounted for).

The Measurement Matrices

As mentioned in the problem formulation, it is assumed that the pursuer measures the position of the evader perpendicular to the reference line and that this measurement is corrupted by noise. Obviously, in addition to the mentioned measurement, quantities like the pursuer's lateral acceleration and angular rates, which are usually necessary for autopilot implementation, are also available to the pursuer (missile). It is assumed that these additional measurements are perfect and decoupled from the evader's position measurement. Under these assumptions, it is easily seen that the general forms of the measurement matrices are

$$H_1 = \left[\begin{array}{cc|ccc} 1 & 0 & 0 & & 0 \\ \hline 0 & 0 & h_{23} & \cdots & h_{2n} \\ 0 & 0 & & & \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & h_{k3} & & h_{kn} \end{array}\right], \quad H_2 = \left[\begin{array}{c} 1 \\ \hline 0 \\ 0 \\ \vdots \\ 0 \end{array}\right]$$

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